# Fakir Mohan Autonomous College, Balasore
## UG - CC- II, Discrete Mathematics

**Notation.** $(\mathbb{Z}_n, +)$ is the ring of integers under addition modulo $n$.

1. Solve the simultaneous congruences:
$$x \equiv 7 \mod 108$$
$$x \equiv 5 \mod 605$$

2. Use the Euclidean algorithm to find the integer $x$ such that $1 = 200x + 641y$. (The integer $x$ is "the inverse of 200 mod 641.")

3. Find the "inverse" of 521 modulo 625. (That is, find an integer $s$ such that $521s \equiv 1 \mod 625$.)

4. Use the Euclidean algorithm to find the GCD of $a = 2^{32} + 1$ and $b = 2^{14} \cdot 5^2 - 1$ and compute the integers $s$ and $t$ such that $\text{GCD}(a, b) = as + bt$. (Fermat, probably around 1640, claimed that $2^{32} + 1$ is prime. Was he correct?)

5. Find the *smallest* pair of integers $a$ and $b$ such that the Euclidean Algorithm, in tabular format, has nine rows. More generally, what is the "worst case" scenario for the Euclidean Algorithm: which pair of integers require the most rows?

6. Use Bezout's Identity to prove Euclid's Lemma: If a prime $p$ divides $ab$ then either $p$ divides $a$ or $p$ divides $b$.

7. Use the Euclidean algorithm (in tabular format) to find the GCD of 11333 and 7213 and compute the integers $s$ and $t$ such that $\text{GCD}(11333, 7213) = 11333s + 7213t$.

8. If $A, B$ are symmetric matrices of same order, prove that $AB - BA$ is a skew symmetric matrix.

9. If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is a skew symmetric matrix, find the value of $a, b,$ and $c$.

10. Express the following matrix as the sum of symmetric and skew symmetric matrix

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}.$$

11. Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $A^2 - 3A - 7I = 0$ .

12. Find the product of matrices $A$ and $B$, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

13. Using elementary row transformation, find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

14. Prove that every square matrix can be uniquely expressed as the sum of a symmetric and a skew symmetric matrix.

15. Prove that Inverse of a square matrix, if it exits, is unique.

Using elementary column transformation, find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

16. Let $M = (a_{ij})$ be an $n \times m$ matrix. Consider a graph $G$ whose vertices are the entries of $M$ and two vertices are adjacent if they lie in the same row or same column. Then

(a) Find the total number of vertices of $G$?
(b) Is $G$ a regular graph? If yes, then find the degree of regularity.
(c) Find the total number of edges of $G$.

17. In any group of $n$ persons ($n \geq 3$), show that there are at least two with the same number of friends.

18. Prove that if a graph has exactly two vertices of odd degree then there must be a path in the graph joining them.

19. Prove or disprove: (i) If every vertex of a simple graph $G$ has degree 2 then $G$ is a cycle.
(ii) A closed trail with all its vertices of degree 2 is a cycle.

20. If $G$ is a simple graph with $n$ vertices and the minimum degree $\delta(G) \geq \frac{n-1}{2}$ then prove that $G$ is connected.

21. Show by means of an example that the condition ( ) ≥ $\frac{n-2}{2}$ for a simple graph $G$, need not imply that $G$ is connected.

Let $G$ be a graph in which there is no pair of adjacent edges. What can you say about the degrees of the vertices in $G$.

22. Let $G$ be a graph with $n$ vertices and $e$ edges. Let $m$ be the smallest positive integer such that $m \geq \frac{2e}{n}$. Prove that $G$ has a vertex of degree at least $m$.

Let $G$ be a graph with $n$ vertices and exactly $n-1$ edges. Prove that $G$ has either a pendant vertex or an isolated vertex.

23. Prove that in a group of six people, there must be three people who are mutually acquainted or three people who are mutually non-acquainted.

24. A simple graph with $n$ vertices and $k$ components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

25. Prove that if a simple graph $G$ is not connected then its complement is connected.

26. Prove that if $v$ is a cut vertex of graph $G$ then $v$ is not a cut vertex of graph $G^c$.

27. Let $G$ be a simple graph with $\delta(G) \geq k$. Show that: (i) $G$ contains a path of length at least $k$.
(ii) If $k \geq 2$ then $G$ contains a cycle of length at least $k+1$.

28. Construct a simple 3-regular graph on 8 vertices that contains no cycle of length 3 (triangle free).

29. Construct a simple 3-regular graph on 10 vertices that contains no induced $C_3$ or $C_4$.