

**Paper Specific Instructions**

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.
2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ)**. Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.
3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ)**. Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.
4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.
5. In all sections, questions not attempted will result in zero mark. In **Section – A (MCQ)**, wrong answer will result in **NEGATIVE** marks. For all 1 mark questions,  $\frac{1}{3}$  marks will be deducted for each wrong answer. For all 2 marks questions,  $\frac{2}{3}$  marks will be deducted for each wrong answer. In **Section – B (MSQ)**, there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C (NAT)** as well.
6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.
7. The Scribble Pad will be provided for rough work.

<b>Special Instructions/Useful Data</b>	
	All angles are in radian
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n): x_i \in \mathbb{R}, 1 \leq i \leq n\}$
$M^T$	Transpose of the matrix $M$
$f'$	Derivative of the function $f$
$P(E)$	Probability of the event $E$
$E(X)$	Expectation of the random variable $X$
$Var(X)$	Variance of the random variable $X$
i.i.d.	Independently and identically distributed
$U(a, b)$	Continuous uniform distribution on $(a, b)$ , $-\infty < a < b < \infty$
$\Phi(a)$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-x^2/2} dx$
$\Gamma(p)$	The gamma function $\Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt, \quad p > 0$
$n!$	The factorial function $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

## SECTION – A

## MULTIPLE CHOICE QUESTIONS (MCQ)

**Q. 1 – Q.10 carry one mark each.**

Q.1 Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers such that  $a_1 = 2$  and, for  $n \geq 1$ ,

$$a_{n+1} = \frac{2a_n + 1}{a_n + 1}.$$

Then

- (A)  $1.5 \leq a_n \leq 2$ , for all natural number  $n \geq 1$
- (B) there exists a natural number  $n \geq 1$  such that  $a_n > 2$
- (C) there exists a natural number  $n \geq 1$  such that  $a_n < 1.5$
- (D) there exists a natural number  $n \geq 1$  such that  $a_n = \frac{1+\sqrt{5}}{2}$

Q.2 The value of

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^{n^2} e^{-2n}$$

is

- (A)  $e^{-2}$
- (B)  $e^{-1}$
- (C)  $e$
- (D)  $e^2$

Q.3 Let  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  be two convergent sequences of real numbers. For  $n \geq 1$ , define  $u_n = \max\{a_n, b_n\}$  and  $v_n = \min\{a_n, b_n\}$ . Then

- (A) neither  $\{u_n\}_{n \geq 1}$  nor  $\{v_n\}_{n \geq 1}$  converges
- (B)  $\{u_n\}_{n \geq 1}$  converges but  $\{v_n\}_{n \geq 1}$  does not converge
- (C)  $\{u_n\}_{n \geq 1}$  does not converge but  $\{v_n\}_{n \geq 1}$  converges
- (D) both  $\{u_n\}_{n \geq 1}$  and  $\{v_n\}_{n \geq 1}$  converge

Q.4

Let  $M = \begin{bmatrix} 1 & 3 \\ 4 & 4 \\ 3 & 2 \\ 1 & 5 \end{bmatrix}$ . If  $I$  is the  $2 \times 2$  identity matrix and  $\mathbf{0}$  is the  $2 \times 2$  zero matrix, then

- (A)  $20M^2 - 13M + 7I = \mathbf{0}$
- (B)  $20M^2 - 13M - 7I = \mathbf{0}$
- (C)  $20M^2 + 13M + 7I = \mathbf{0}$
- (D)  $20M^2 + 13M - 7I = \mathbf{0}$

Q.5 Let  $X$  be a random variable with the probability density function

$$f(x) = \begin{cases} \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}, & x \geq 0, \alpha > 0, p > 0, \\ 0, & \text{otherwise.} \end{cases}$$

If  $E(X) = 20$  and  $Var(X) = 10$ , then  $(\alpha, p)$  is

- (A) (2, 20)                      (B) (2, 40)                      (C) (4, 20)                      (D) (4, 40)

Q.6 Let  $X$  be a random variable with the distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4} + \frac{4x - x^2}{8}, & 0 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

Then

$$P(X = 0) + P(X = 1.5) + P(X = 2) + P(X \geq 1)$$

equals

- (A)  $\frac{3}{8}$                       (B)  $\frac{5}{8}$                       (C)  $\frac{7}{8}$                       (D) 1

Q.7 Let  $X_1, X_2$  and  $X_3$  be i.i.d.  $U(0, 1)$  random variables. Then  $E\left(\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right)$  equals

- (A)  $\frac{1}{3}$                       (B)  $\frac{1}{2}$                       (C)  $\frac{2}{3}$                       (D)  $\frac{3}{4}$

Q.8 Let  $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$  and  $x_5 = 0$  be the observed values of a random sample of size 5 from a discrete distribution with the probability mass function

$$f(x; \theta) = P(X = x) = \begin{cases} \frac{\theta}{3}, & x = 0, \\ \frac{2\theta}{3}, & x = 1, \\ \frac{1 - \theta}{2}, & x = 2, 3, \end{cases}$$

where  $\theta \in [0, 1]$  is the unknown parameter. Then the maximum likelihood estimate of  $\theta$  is

- (A)  $\frac{2}{5}$                       (B)  $\frac{3}{5}$                       (C)  $\frac{5}{7}$                       (D)  $\frac{5}{9}$

Q.9 Consider four coins labelled as 1, 2, 3 and 4. Suppose that the probability of obtaining a 'head' in a single toss of the  $i^{\text{th}}$  coin is  $\frac{i}{4}$ ,  $i = 1, 2, 3, 4$ . A coin is chosen uniformly at random and flipped. Given that the flip resulted in a 'head', the conditional probability that the coin was labelled either 1 or 2 equals

(A)  $\frac{1}{10}$

(B)  $\frac{2}{10}$

(C)  $\frac{3}{10}$

(D)  $\frac{4}{10}$

Q.10 Consider the linear regression model  $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ ;  $i = 1, 2, \dots, n$ , where  $\epsilon_i$ 's are i.i.d. standard normal random variables. Given that

$$\frac{1}{n} \sum_{i=1}^n x_i = 3.2, \quad \frac{1}{n} \sum_{i=1}^n y_i = 4.2, \quad \frac{1}{n} \sum_{j=1}^n \left( x_j - \frac{1}{n} \sum_{i=1}^n x_i \right)^2 = 1.5 \text{ and}$$

$$\frac{1}{n} \sum_{j=1}^n \left( x_j - \frac{1}{n} \sum_{i=1}^n x_i \right) \left( y_j - \frac{1}{n} \sum_{i=1}^n y_i \right) = 1.7,$$

the maximum likelihood estimates of  $\beta_0$  and  $\beta_1$ , respectively, are

(A)  $\frac{17}{15}$  and  $\frac{32}{75}$

(B)  $\frac{32}{75}$  and  $\frac{17}{15}$

(C)  $\frac{17}{15}$  and  $\frac{43}{75}$

(D)  $\frac{43}{75}$  and  $\frac{17}{15}$

**Q. 11 – Q. 30 carry two marks each.**

Q.11 Let  $f: [-1, 1] \rightarrow \mathbb{R}$  be defined by  $f(x) = \frac{x^2 + [\sin \pi x]}{1 + |x|}$ , where  $[y]$  denotes the greatest integer less than or equal to  $y$ . Then

- (A)  $f$  is continuous at  $-\frac{1}{2}, 0, 1$
- (B)  $f$  is discontinuous at  $-1, 0, \frac{1}{2}$
- (C)  $f$  is discontinuous at  $-1, -\frac{1}{2}, 0, \frac{1}{2}$
- (D)  $f$  is continuous everywhere except at 0

Q.12 Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2 - \frac{\cos x}{2}$  and  $g(x) = \frac{x \sin x}{2}$ . Then

- (A)  $f(x) = g(x)$  for more than two values of  $x$
- (B)  $f(x) \neq g(x)$ , for all  $x$  in  $\mathbb{R}$
- (C)  $f(x) = g(x)$  for exactly one value of  $x$
- (D)  $f(x) = g(x)$  for exactly two values of  $x$

Q.13 Consider the domain  $D = \{ (x, y) \in \mathbb{R}^2: x \leq y \}$  and the function  $h: D \rightarrow \mathbb{R}$  defined by

$$h((x, y)) = (x - 2)^4 + (y - 1)^4, (x, y) \in D.$$

Then the minimum value of  $h$  on  $D$  equals

- (A)  $\frac{1}{2}$
- (B)  $\frac{1}{4}$
- (C)  $\frac{1}{8}$
- (D)  $\frac{1}{16}$

Q.14 Let  $M = [X \ Y \ Z]$  be an orthogonal matrix with  $X, Y, Z \in \mathbb{R}^3$  as its column vectors. Then

$$Q = X X^T + Y Y^T$$

- (A) is a skew-symmetric matrix
- (B) is the  $3 \times 3$  identity matrix
- (C) satisfies  $Q^2 = Q$
- (D) satisfies  $QZ = Z$

Q.15 Let  $f: [0, 3] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ e^{x^2} - e, & 1 \leq x < 2 \\ e^{x^2} + 1, & 2 \leq x \leq 3. \end{cases}$$

Now, define  $F: [0, 3] \rightarrow \mathbb{R}$  by

$$F(0) = 0 \text{ and } F(x) = \int_0^x f(t)dt, \text{ for } 0 < x \leq 3.$$

Then

- (A)  $F$  is differentiable at  $x = 1$  and  $F'(1) = 0$
- (B)  $F$  is differentiable at  $x = 2$  and  $F'(2) = 0$
- (C)  $F$  is not differentiable at  $x = 1$
- (D)  $F$  is differentiable at  $x = 2$  and  $F'(2) = 1$

Q.16 If  $x, y$  and  $z$  are real numbers such that  $4x + 2y + z = 31$  and  $2x + 4y - z = 19$ , then the value of  $9x + 7y + z$

- (A) cannot be computed from the given information
- (B) equals  $\frac{281}{3}$
- (C) equals  $\frac{182}{3}$
- (D) equals  $\frac{218}{3}$

Q.17 Let  $M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ . If

$$V = \left\{ (x, y, 0) \in \mathbb{R}^3 : M \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \text{ and } W = \left\{ (x, y, z) \in \mathbb{R}^3 : M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\},$$

then

- (A) the dimension of  $V$  equals 2
- (B) the dimension of  $W$  equals 2
- (C) the dimension of  $V$  equals 1
- (D)  $V \cap W = \{(0,0,0)\}$

Q.18 Let  $M$  be a  $3 \times 3$  non-zero, skew-symmetric real matrix. If  $I$  is the  $3 \times 3$  identity matrix, then

- (A)  $M$  is invertible
- (B) the matrix  $I + M$  is invertible
- (C) there exists a non-zero real number  $\alpha$  such that  $\alpha I + M$  is not invertible
- (D) all the eigenvalues of  $M$  are real

Q.19 Let  $X$  be a random variable with the moment generating function

$$M_X(t) = \frac{6}{\pi^2} \sum_{n \geq 1} \frac{e^{t^2/2n}}{n^2}, \quad t \in \mathbb{R}.$$

Then  $P(X \in \mathbb{Q})$ , where  $\mathbb{Q}$  is the set of rational numbers, equals

- (A) 0
- (B)  $\frac{1}{4}$
- (C)  $\frac{1}{2}$
- (D)  $\frac{3}{4}$

Q.20 Let  $X$  be a discrete random variable with the moment generating function

$$M_X(t) = \frac{(1 + 3e^t)^2(3 + e^t)^3}{1024}, \quad t \in \mathbb{R}.$$

Then

- (A)  $E(X) = \frac{9}{4}$
- (B)  $Var(X) = \frac{15}{32}$
- (C)  $P(X \geq 1) = \frac{27}{1024}$
- (D)  $P(X = 5) = \frac{3}{1024}$

Q.21 Let  $\{X_n\}_{n \geq 1}$  be a sequence of independent random variables with  $X_n$  having the probability density function as

$$f_n(x) = \begin{cases} \frac{1}{2^{n/2} \Gamma(\frac{n}{2})} e^{-x/2} x^{(n/2)-1}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lim_{n \rightarrow \infty} \left[ P\left(X_n > \frac{3}{4}n\right) + P\left(X_n > n + 2\sqrt{2n}\right) \right]$$

equals

- (A)  $1 + \Phi(2)$
- (B)  $1 - \Phi(2)$
- (C)  $\Phi(2)$
- (D)  $2 - \Phi(2)$



Q.22 Let  $X$  be a Poisson random variable with mean  $\frac{1}{2}$ . Then  $E((X + 1)!)$  equals

- (A)  $2 e^{-\frac{1}{2}}$                       (B)  $4 e^{-\frac{1}{2}}$                       (C)  $4 e^{-1}$                       (D)  $2 e^{-1}$

Q.23 Let  $X$  be a standard normal random variable. Then  $P(X^3 - 2X^2 - X + 2 > 0)$  equals

- (A)  $2\Phi(1) - 1$     (B)  $1 - \Phi(2)$   
 (C)  $2\Phi(1) - \Phi(2)$     (D)  $\Phi(2) - \Phi(1)$

Q.24 Let  $X$  and  $Y$  have the joint probability density function

$$f(x, y) = \begin{cases} 2, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $a = E(Y|X = \frac{1}{2})$  and  $b = Var(Y|X = \frac{1}{2})$ . Then  $(a, b)$  is

- (A)  $(\frac{3}{4}, \frac{7}{12})$     (B)  $(\frac{1}{4}, \frac{1}{48})$   
 (C)  $(\frac{1}{4}, \frac{7}{12})$     (D)  $(\frac{3}{4}, \frac{1}{48})$

Q.25 Let  $X$  and  $Y$  have the joint probability mass function

$$P(X = m, Y = n) = \begin{cases} \frac{m+n}{21}, & m = 1, 2, 3; n = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $P(X = 2|Y = 2)$  equals

- (A)  $\frac{1}{3}$     (B)  $\frac{2}{3}$     (C)  $\frac{1}{2}$     (D)  $\frac{1}{4}$

Q.26 Let  $X$  and  $Y$  be two independent standard normal random variables. Then the probability density function of  $Z = \frac{|X|}{|Y|}$  is

- (A)  $f(z) = \begin{cases} \frac{\sqrt{1/2}}{\sqrt{\pi}} e^{-\frac{z}{2}} z^{-\frac{1}{2}}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$                       (B)  $f(z) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-z^2/2}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$   
 (C)  $f(z) = \begin{cases} e^{-z}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$                       (D)  $f(z) = \begin{cases} \frac{2}{\pi} \frac{1}{(1+z^2)}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$

Q.27 Let  $X$  and  $Y$  have the joint probability density function

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then the correlation coefficient between  $X$  and  $Y$  equals

- (A)  $\frac{1}{3}$                       (B)  $\frac{1}{\sqrt{3}}$                       (C)  $\frac{1}{\sqrt{2}}$                       (D)  $\frac{2}{\sqrt{3}}$

Q.28 Let  $x_1 = -2, x_2 = 1$  and  $x_3 = -1$  be the observed values of a random sample of size three from a discrete distribution with the probability mass function

$$f(x; \theta) = P(X = x) = \begin{cases} \frac{1}{2\theta + 1}, & x \in \{-\theta, -\theta + 1, \dots, 0, \dots, \theta\}, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta \in \Theta = \{1, 2, \dots\}$  is the unknown parameter. Then the method of moment estimate of  $\theta$  is

- (A) 1                      (B) 2                      (C) 3                      (D) 4

Q.29 Let  $X$  be a random sample from a discrete distribution with the probability mass function

$$f(x; \theta) = P(X = x) = \begin{cases} \frac{1}{\theta}, & x = 1, 2, \dots, \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta \in \Theta = \{20, 40\}$  is the unknown parameter. Consider testing

$$H_0: \theta = 40 \text{ against } H_1: \theta = 20$$

at a level of significance  $\alpha = 0.1$ . Then the uniformly most powerful test rejects  $H_0$  if and only if

- (A)  $X \leq 4$                       (B)  $X > 4$   
 (C)  $X \geq 3$                       (D)  $X < 3$

Q.30 Let  $X_1$  and  $X_2$  be a random sample of size 2 from a discrete distribution with the probability mass function

$$f(x; \theta) = P(X = x) = \begin{cases} \theta, & x = 0, \\ 1 - \theta, & x = 1, \end{cases}$$

where  $\theta \in \Theta = \{0.2, 0.4\}$  is the unknown parameter. For testing  $H_0: \theta = 0.2$  against  $H_1: \theta = 0.4$ , consider a test with the critical region

$$C = \{(x_1, x_2) \in \{0, 1\} \times \{0, 1\} : x_1 + x_2 < 2\}.$$

Let  $\alpha$  and  $\beta$  denote the probability of Type I error and power of the test, respectively.

Then  $(\alpha, \beta)$  is

- (A) (0.36, 0.74)                      (B) (0.64, 0.36)  
 (C) (0.05, 0.64)                      (D) (0.36, 0.64)

**SECTION - B****MULTIPLE SELECT QUESTIONS (MSQ)**

**Q. 31 – Q. 40 carry two marks each.**

Q.31 Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers such that

$$a_n = \sum_{k=n+1}^{2n} \frac{1}{k}, \quad n \geq 1.$$

Then which of the following statement(s) is (are) true?

- (A)  $\{a_n\}_{n \geq 1}$  is an increasing sequence
- (B)  $\{a_n\}_{n \geq 1}$  is bounded below
- (C)  $\{a_n\}_{n \geq 1}$  is bounded above
- (D)  $\{a_n\}_{n \geq 1}$  is a convergent sequence

Q.32 Let  $\sum_{n \geq 1} a_n$  be a convergent series of positive real numbers. Then which of the following statement(s) is (are) true?

- (A)  $\sum_{n \geq 1} (a_n)^2$  is always convergent
- (B)  $\sum_{n \geq 1} \sqrt{a_n}$  is always convergent
- (C)  $\sum_{n \geq 1} \frac{\sqrt{a_n}}{n}$  is always convergent
- (D)  $\sum_{n \geq 1} \frac{\sqrt{a_n}}{n^{1/4}}$  is always convergent

Q.33 Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers such that  $a_1 = 3$  and, for  $n \geq 1$ ,

$$a_{n+1} = \frac{a_n^2 - 2a_n + 4}{2}.$$

Then which of the following statement(s) is (are) true?

- (A)  $\{a_n\}_{n \geq 1}$  is a monotone sequence
- (B)  $\{a_n\}_{n \geq 1}$  is a bounded sequence
- (C)  $\{a_n\}_{n \geq 1}$  does not have finite limit, as  $n \rightarrow \infty$
- (D)  $\lim_{n \rightarrow \infty} a_n = 2$

Q.34 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x^4(2 + \sin \frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then which of the following statement(s) is (are) true?

- (A)  $f$  attains its minimum at 0
- (B)  $f$  is monotone
- (C)  $f$  is differentiable at 0
- (D)  $f(x) > 2x^4 + x^3$ , for all  $x > 0$

Q.35 Let  $P$  be a probability function that assigns the same weight to each of the points of the sample space  $\Omega = \{1,2,3,4\}$ . Consider the events  $E = \{1,2\}$ ,  $F = \{1,3\}$  and  $G = \{3,4\}$ . Then which of the following statement(s) is (are) true?

- (A)  $E$  and  $F$  are independent
- (B)  $E$  and  $G$  are independent
- (C)  $F$  and  $G$  are independent
- (D)  $E, F$  and  $G$  are independent

Q.36 Let  $X_1, X_2, \dots, X_n, n \geq 5$ , be a random sample from a distribution with the probability density function

$$f(x; \theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta \in \mathbb{R}$  is the unknown parameter. Then which of the following statement(s) is (are) true?

- (A) A 95% confidence interval of  $\theta$  has to be of finite length
- (B)  $(\min\{X_1, X_2, \dots, X_n\} + \frac{1}{n} \ln(0.05), \min\{X_1, X_2, \dots, X_n\})$  is a 95% confidence interval of  $\theta$
- (C) A 95% confidence interval of  $\theta$  can be of length 1
- (D) A 95% confidence interval of  $\theta$  can be of length 2

Q.37 Let  $X_1, X_2, \dots, X_n$  be a random sample from  $U(0, \theta)$ , where  $\theta > 0$  is the unknown parameter. Let  $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$ . Then which of the following is (are) consistent estimator(s) of  $\theta^3$ ?

- (A)  $8X_n^3$
- (B)  $X_{(n)}^3$
- (C)  $\left(\frac{2}{n} \sum_{i=5}^n X_i\right)^3$
- (D)  $\frac{nX_{(n)}^3 + 1}{n+1}$

Q.38 Let  $X_1, X_2, \dots, X_n$  be a random sample from a distribution with the probability density function

$$f(x; \theta) = \begin{cases} c(\theta) e^{-(x-\theta)}, & x \geq 2\theta, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta \in \mathbb{R}$  is the unknown parameter. Then which of the following statement(s) is (are) true?

(A) The maximum likelihood estimator of  $\theta$  is  $\frac{\min\{X_1, X_2, \dots, X_n\}}{2}$

(B)  $c(\theta) = 1$ , for all  $\theta \in \mathbb{R}$

(C) The maximum likelihood estimator of  $\theta$  is  $\min\{X_1, X_2, \dots, X_n\}$

(D) The maximum likelihood estimator of  $\theta$  does not exist

Q.39 Let  $X_1, X_2, \dots, X_n$  be a random sample from a distribution with the probability density function

$$f(x; \theta) = \begin{cases} \theta^2 x e^{-\theta x}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta > 0$  is the unknown parameter. If  $Y = \sum_{i=1}^n X_i$ , then which of the following statement(s) is (are) true?

(A)  $Y$  is a complete sufficient statistic for  $\theta$

(B)  $\frac{2n}{Y}$  is the uniformly minimum variance unbiased estimator of  $\theta$

(C)  $\frac{2n-1}{Y}$  is the uniformly minimum variance unbiased estimator of  $\theta$

(D)  $\frac{2n+1}{Y}$  is the uniformly minimum variance unbiased estimator of  $\theta$

Q.40 Let  $X_1, X_2, \dots, X_n$  be a random sample from  $U(\theta, \theta + 1)$ , where  $\theta \in \mathbb{R}$  is the unknown parameter.

Let  $U = \max\{X_1, X_2, \dots, X_n\}$  and  $V = \min\{X_1, X_2, \dots, X_n\}$ . Then which of the following statement(s) is (are) true?

(A)  $U$  is a consistent estimator of  $\theta$

(B)  $V$  is a consistent estimator of  $\theta$

(C)  $2U - V - 2$  is a consistent estimator of  $\theta$

(D)  $2V - U + 1$  is a consistent estimator of  $\theta$

## SECTION – C

## NUMERICAL ANSWER TYPE (NAT)

**Q. 41 – Q. 50 carry one mark each.**

Q.41 Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers such that

$$a_n = \frac{1 + 3 + 5 + \dots + (2n - 1)}{n!}, \quad n \geq 1.$$

Then  $\sum_{n \geq 1} a_n$  converges to \_\_\_\_\_

Q.42 Let

$$S = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, \quad \sqrt{4 - (x - 2)^2} \leq y \leq \sqrt{9 - (x - 3)^2}\}.$$

Then the area of  $S$  equals \_\_\_\_\_

Q.43 Let  $S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ . Then the area of  $S$  equals \_\_\_\_\_

Q.44 Let

$$J = \frac{1}{\pi} \int_0^1 t^{-\frac{1}{2}} (1 - t)^{\frac{3}{2}} dt.$$

Then the value of  $J$  equals \_\_\_\_\_

Q.45 A fair die is rolled three times independently. Given that 6 appeared at least once, the conditional probability that 6 appeared exactly twice equals \_\_\_\_\_

Q.46 Let  $X$  and  $Y$  be two positive integer valued random variables with the joint probability mass function

$$P(X = m, Y = n) = \begin{cases} g(m) h(n), & m, n \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $g(m) = \left(\frac{1}{2}\right)^{m-1}$ ,  $m \geq 1$  and  $h(n) = \left(\frac{1}{3}\right)^n$ ,  $n \geq 1$ . Then  $E(XY)$  equals \_\_\_\_\_

Q.47 Let  $E, F$  and  $G$  be three events such that

$$P(E \cap F \cap G) = 0.1, P(G|F) = 0.3 \text{ and } P(E|F \cap G) = P(E|F).$$

Then  $P(G|E \cap F)$  equals \_\_\_\_\_

Q.48 Let  $A_1, A_2$  and  $A_3$  be three events such that

$$P(A_i) = \frac{1}{3}, i = 1, 2, 3; P(A_i \cap A_j) = \frac{1}{6}, 1 \leq i \neq j \leq 3 \text{ and } P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}.$$

Then the probability that none of the events  $A_1, A_2, A_3$  occur equals \_\_\_\_\_

Q.49 Let  $X_1, X_2, \dots, X_n$  be a random sample from the distribution with the probability density function

$$f(x) = \frac{1}{4} e^{-|x-4|} + \frac{1}{4} e^{-|x-6|}, x \in \mathbb{R}.$$

Then  $\frac{1}{n} \sum_{i=1}^n X_i$  converges in probability to \_\_\_\_\_

Q.50 Let  $x_1 = 1.1, x_2 = 2.2$  and  $x_3 = 3.3$  be the observed values of a random sample of size three from a distribution with the probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta \in \Theta = \{1, 2, \dots\}$  is the unknown parameter. Then the maximum likelihood estimate of  $\theta$  equals \_\_\_\_\_

**Q. 51 – Q. 60 carry two marks each.**

Q.51 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function such that  $f'$  is continuous on  $\mathbb{R}$  with  $f'(3) = 18$ . Define

$$g_n(x) = n \left( f\left(x + \frac{5}{n}\right) - f\left(x - \frac{2}{n}\right) \right).$$

Then  $\lim_{n \rightarrow \infty} g_n(3)$  equals \_\_\_\_\_

Q.52 Let  $M = \sum_{i=1}^4 X_i X_i^T$ , where

$$X_1^T = [1 \quad -1 \quad 1 \quad 0], X_2^T = [1 \quad 1 \quad 0 \quad 1], X_3^T = [1 \quad 3 \quad 1 \quad 0] \text{ and } X_4^T = [1 \quad 1 \quad 1 \quad 0].$$

Then the rank of  $M$  equals \_\_\_\_\_

Q.53 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function with  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} f'(x) = 2$ . Then

$$\lim_{x \rightarrow \infty} \left( 1 + \frac{f(x)}{x^2} \right)^x$$

equals \_\_\_\_\_

Q.54 The value of

$$\int_0^{\frac{\pi}{2}} \left( \int_0^x e^{\sin y} \sin x \, dy \right) dx$$

equals \_\_\_\_\_

Q.55 Let  $X$  be a random variable with the probability density function

$$f(x) = \begin{cases} 4x^k, & 0 < x < 1, \\ x - \frac{x^2}{2}, & 1 \leq x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

where  $k$  is a positive integer. Then  $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$  equals \_\_\_\_\_



Q.56 Let  $X$  and  $Y$  be two discrete random variables with the joint moment generating function

$$M_{X,Y}(t_1, t_2) = \left(\frac{1}{3} e^{t_1} + \frac{2}{3}\right)^2 \left(\frac{2}{3} e^{t_2} + \frac{1}{3}\right)^3, t_1, t_2 \in \mathbb{R}.$$

Then  $P(2X + 3Y > 1)$  equals \_\_\_\_\_

Q.57 Let  $X_1, X_2, X_3$  and  $X_4$  be i.i.d. discrete random variables with the probability mass function

$$P(X_1 = n) = \begin{cases} \frac{3^{n-1}}{4^n}, & n = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $P(X_1 + X_2 + X_3 + X_4 = 6)$  equals \_\_\_\_\_

Q.58 Let  $X$  be a random variable with the probability mass function

$$P(X = n) = \begin{cases} \frac{1}{10}, & n = 1, 2, \dots, 10, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $E(\max\{X, 5\})$  equals \_\_\_\_\_

Q.59 Let  $X$  be a sample observation from  $U(\theta, \theta^2)$  distribution, where  $\theta \in \Theta = \{2, 3\}$  is the unknown parameter. For testing

$$H_0: \theta = 2 \text{ against } H_1: \theta = 3,$$

let  $\alpha$  and  $\beta$  be the size and power, respectively, of the test that rejects  $H_0$  if and only if  $X \geq 3.5$ .

Then  $\alpha + \beta$  equals \_\_\_\_\_

Q.60 A fair die is rolled four times independently. For  $i = 1, 2, 3, 4$ , define

$$Y_i = \begin{cases} 1, & \text{if 6 appears in the } i^{\text{th}} \text{ throw,} \\ 0, & \text{otherwise.} \end{cases}$$

Then  $P(\max\{Y_1, Y_2, Y_3, Y_4\} = 1)$  equals \_\_\_\_\_

**END OF THE QUESTION PAPER**

Paper Code : MS

Q No	Question Type (QT)	Section	Key/Range
1	MCQ	A	A
2	MCQ	A	A
3	MCQ	A	D
4	MCQ	A	B
5	MCQ	A	B
6	MCQ	A	C
7	MCQ	A	C
8	MCQ	A	B
9	MCQ	A	C
10	MCQ	A	D
11	MCQ	A	B
12	MCQ	A	D
13	MCQ	A	C
14	MCQ	A	C
15	MCQ	A	A
16	MCQ	A	D
17	MCQ	A	C
18	MCQ	A	B
19	MCQ	A	A
20	MCQ	A	A
21	MCQ	A	D
22	MCQ	A	B
23	MCQ	A	C

Paper Code : MS

Q No	Question Type (QT)	Section	Key/Range
24	MCQ	A	D
25	MCQ	A	A
26	MCQ	A	D
27	MCQ	A	C
28	MCQ	A	B
29	MCQ	A	A
30	MCQ	A	D
31	MSQ	B	A;B;C;D
32	MSQ	B	A;C
33	MSQ	B	A;C
34	MSQ	B	A;C
35	MSQ	B	A;C
36	MSQ	B	B;C;D
37	MSQ	B	B;C;D
38	MSQ	B	A
39	MSQ	B	A;C
40	MSQ	B	B;C;D
41	NAT	C	5.40 to 5.50
42	NAT	C	7.80 to 7.90
43	NAT	C	1.90 to 2.10
44	NAT	C	0.35 to 0.40
45	NAT	C	0.16 to 0.17
46	NAT	C	2.50 to 3.50

Paper Code : MS

Q No	Question Type (QT)	Section	Key/Range
47	NAT	C	0.25 to 0.35
48	NAT	C	0.30 to 0.40
49	NAT	C	4.75 to 5.25
50	NAT	C	1.90 to 2.10
51	NAT	C	125 to 127
52	NAT	C	3.0 to 3.0
53	NAT	C	7.25 to 7.50
54	NAT	C	1.70 to 1.80
55	NAT	C	0.85 to 0.90
56	NAT	C	0.97 to 0.99
57	NAT	C	0.01 to 0.03
58	NAT	C	6.25 to 6.75
59	NAT	C	1.10 to 1.20
60	NAT	C	0.50 to 0.53